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p-Sidon Sets

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Let G be a compact Abelian group with character group X. Bożejko and Pytlik [*Colloq. Math.* **25** (1972), 117–124] introduced and studied several special types of lacunary subsets of X. This paper is based upon a hitherto unpublished detailed study of those types that most resemble Sidon sets, which the present authors had independently introduced and studied under the name of p-Sidon sets. Some, but not all, aspects of the theory of Sidon (= 1-Sidon) sets carry over to the more general setting. In Section 1 some properties of sets equivalent to p-Sidonicity are given. Section 2 contains several useful consequences of p-Sidonicity; see Theorems 2.1 and 2.4 and Corollaries 2.6 and 2.7. In Section 3, it is shown that certain Λ_q sets also satisfy some of the consequences listed in Section 2. Nevertheless, Λ_q sets need not be p-Sidon sets; see Theorem 3.1. Examples of (4/3)-Sidon sets that are not Sidon sets are given in Section 5. The proof that these sets are (4/3)-Sidon sets requires a brief study of 4-norms in Varopoulos algebras; see Section 4. In Section 6, some special results for the circle group are deduced. Many of these results appear to be new even for $p = 1$.

1. DEFINITIONS AND EQUIVALENCES

Throughout this paper, G denotes a (Hausdorff) compact Abelian group and X its character group. Except in 3.1, p denotes a real number, $1 \leqslant p < 2$. $\mathfrak{T} = \mathfrak{T}(G)$ denotes the space of complex-valued trigonometric polynomials on G; $C = C(G)$ denotes the space of

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complex-valued continuous functions on G; $L^r = L^r(G)$ denotes the Lebesgue space of complex-valued functions on G constructed relative to normalised Haar measure λ on G; $\mathbf{M} = \mathbf{M}(G)$ denotes the space of complex Radon measures on G.

$l^r(E)$ has its usual meaning for any set E, and $l^r = l^r(X)$. If $p \in [1, 2)$, p' denotes $p/(p-1)$ ($=\infty$ if $p = 1$). For typographical reasons, we write $a = 2p/(3p-2)$, $b = 2(p-1)/(2-p)$; note that $1 < a \leqslant 2$ and that $a' = 2p/(2-p)$.

If $E \subseteq X$ and $F \subseteq \mathbf{M}$, $F^\wedge$ denotes the Fourier image of F and F_E the set of E-spectral elements of F.

DEFINITION 1.1. A set $E \subseteq X$ is said to be p-Sidon if and only if $\sup\{\|f^\wedge\|_p : f \in \mathfrak{T}_E, \|f\|_\infty = 1\} < \infty$. In the terminology of [3], this means precisely that E belongs to class $S_{\infty,p}$.

Note that E is 1-Sidon if and only if it is Sidon. Most of the results given in this paper are known for $p = 1$, but see Section 6.

In the sequel, $\kappa = \kappa(E, p)$ denotes a nonnegative real number possibly depending upon E and p, not necessarily the same in different places.

THEOREM 1.2. *Let $E \subseteq X$. The following conditions are two-by-two equivalent*:

(i) *E is p-Sidon*;

(ii) *there exists κ such that*

$$\|f^\wedge\|_p \leqslant \kappa \|f\|_\infty \tag{1.2.1}$$

for $f \in \mathfrak{T}_E$ (or C_E or L^∞_E);

(iii) $C_E{}^\wedge \subseteq l^p$ *(or* $(L^\infty_E)^\wedge \subseteq l^p$*)*;

(iv) $\mathbf{M}^\wedge \mid E \supseteq l^{p'}(E)$;

(v) *there exists κ such that every $\psi \in l^{p'}(E)$ (or $c_0(E)$ if $p = 1$) is $f^\wedge \mid E$ for some $f \in L^1$ satisfying*

$$\|f\|_1 \leqslant \kappa \|\psi\|_{p'}. \tag{1.2.2}$$

For the proof of this theorem, see [3, Theorem 1] and [10, Theorem 4].

The known case $p = 1$ of the next theorem ([9, 2]) extends at once to $1 \leqslant p < 2$, and we omit the proof. This result was independently observed by Hahn [10]; he also omitted the proof.

THEOREM 1.3. *Let $E \subseteq X$, and let $w \in c_0^+(X)$. In order that E be p-Sidon, it is necessary and sufficient that $f^\wedge \in l^p$ for every $f \in C_E(G)$ such that*

$$\sum_X |f^\wedge|^r < \infty \qquad \text{for all} \quad r > p$$

and

$$\sum_X w\, |f^\wedge|^p < \infty.$$

Remark 1.4. If E is p-Sidon, then by 1.2(iii)

$$C_E^\wedge \mid E \subseteq l^p(E). \tag{1.4.1}$$

If $p = 1$, we can sharpen (1.4.1) into equality. This sharpening is not possible in general for $p > 1$. In fact, if equality holds in (1.4.1) for some $p > 1$, then E must be finite. This is a corollary of the following lemma.

LEMMA 1.5. *Let S be a closed translation-invariant linear subspace of C such that $S^\wedge$ is stable under multiplication by ± 1-valued functions on X. Then $S \cap X$ is a Sidon subset of X and $S = C_{S \cap X}$.*

Proof. By (38.22.b) of [11], S is a closed ideal in C and so, by (38.7) of [11], we have $S = C_{S \cap X}$. Now consider any $f \in C_{S \cap X}$. For every ± 1-valued function ω on X, we have by hypothesis that $\omega f^\wedge$ belongs to $S^\wedge \subseteq (L^\infty)^\wedge$. Hence, by Theorem (3.1) in [5], $f^\wedge$ belongs to $l^1(X)$. That is, $S \cap X$ is a Sidon set.

2. CONSEQUENCES OF p-SIDONICITY

Recall that $1 \leqslant p < 2$, $a = 2p/(3p-2)$, and $a' = 2p/(2-p)$. The next theorem is Theorem 2(i) in [3]. We give a different and very short proof.

THEOREM 2.1. *If $E \subseteq X$ is p-Sidon, then there exists κ such that*

$$\| \nu^\wedge \|_{a'} \leqslant \kappa \| \nu \| \qquad \text{for all} \quad \nu \in \mathbf{M}_E. \tag{2.1.1}$$

Proof. The hypothesis entails that $(\nu * f)^\wedge = \nu^\wedge f^\wedge \in l^p$ for every $f \in C$. Hence, by Corollary (2.3) in [5], we have $\nu^\wedge \in l^{a'}$. The rest follows from the closed graph theorem.

Remarks 2.2. (i) If κ is as in (1.2.1), the constant in (2.1.1) can be taken to be $2^{1/2}\kappa$. Further, if E has cardinal $\nu(E) \geqslant 2$, the best constant in (2.1.1) is at least $2^{1/a'}\pi/4$, which is greater than 1 if p is close to 1.

(ii) As will appear in 3.2, (2.1.1) holds if E is a Λ_a set. In view of 3.1 below, this shows that (2.1.1) does not imply that E is p-Sidon (when $p < 2$).

THEOREM 2.3. *Condition* (2.1.1) *is equivalent to each of the following conditions*:

(i) $(L^\infty)^\wedge \mid E \supseteq l^a(E)$;

(ii) *there exists a* κ *such that every* $\psi \in l^a(E)$ *is* $h^\wedge \mid E$ *for some* $h \in C$ *satisfying* $\| h \|_\infty \leqslant \kappa \| \psi \|_a$.

Hence (i) *and* (ii) *are true whenever E is p-Sidon.*

Proof. (a) First we show that (2.1.1) implies (i). If (2.1.1) holds, and if $\psi \in l^a(E)$ and $f \in L^1_E$, then $f^\wedge\psi \in l^1(E)$. Hence

$$f \mapsto \sum_E f^\wedge \psi$$

is a continuous linear functional on L^1_E. Hence there exists an $h \in L^\infty$ such that

$$\sum_E f^\wedge \psi = f * h(e) \qquad \text{for all} \quad f \in L^1_E .$$

(b) The proof that (i) implies (ii) is very similar to the proof that (iii) implies (i) in Theorem 1 of [3]; we omit the details.

(c) Finally, we show that (ii) implies (2.1.1). Assume (ii) and suppose that $\nu \in \mathbf{M}_E$ and $\psi \in l^a(E)$. By (ii), $\psi = h^\wedge \mid E$ for some $h \in C$ satisfying

$$\| h \|_\infty \leqslant \kappa \| \psi \|_a . \tag{2.3.1}$$

Let (K_j) be an approximate identity in $\mathfrak{T}$ such that $\| K_j \|_1 \leqslant 1$. From (2.3.1) we obtain

$$\left| \sum_E \nu^\wedge \psi K_j{}^\wedge \right| = \left| \sum_E \nu^\wedge h^\wedge K_j{}^\wedge \right| = | \nu * K_j * h(e) |$$
$$\leqslant \| \nu \| \| K_j * h \|_\infty \leqslant \| \nu \| \kappa \| \psi \|_a .$$

By adjusting the arguments of the values of ψ, this entails that

$$\sum_E | \nu^\wedge \psi K_j{}^\wedge | \leqslant \kappa \| \nu \| \| \psi \|_a ,$$

and hence

$$\| \nu^\wedge K_j^\wedge \|_{a'} \leqslant \kappa \| \nu \|. \tag{2.3.2}$$

Letting j vary, (2.3.2) leads to (2.1.1). ∎

If $p > 1$ and $E \subseteq X$ is infinite, it is false to assert that $C^\wedge \mid E \subseteq l^a(E)$. In fact, $l^2(E_1) \subseteq C^\wedge \mid E_1$ for all infinite Sidon subsets E_1 of E.

The next theorem has several interesting corollaries.

THEOREM 2.4. *Let $E \subseteq X$. The following conditions are equivalent, and they both hold if E is a p-Sidon set.*

(i) *There exists κ such that if $\mu \in \mathbf{M}_E$ and $\mu^\wedge \in l^a$, then $\mu \in L^r$ and*

$$\| \mu \|_r \leqslant \kappa r^{1/2} \| \mu^\wedge \|_a \tag{2.4.1}$$

for all $r \in [1, \infty)$.

(ii) *There exists κ such that*

$$\Big(\sum_E | g^\wedge |^{a'} \Big)^{1/a'} \leqslant \kappa r'^{1/2} \| g \|_r \tag{2.4.2}$$

for all $g \in L^r$ and $r \in (1, \infty]$.

Proof. Let E be p-Sidon. According to Eq. (9) in [3], there exists κ such that

$$\| f \|_r \leqslant \kappa r^{1/2} \| f^\wedge \|_a \tag{2.4.3}$$

for all $f \in \mathfrak{T}_E$ and all $r \in [1, \infty)$. Now consider $\mu \in \mathbf{M}_E$ such that $\mu^\wedge \in l^a$, and let (K_j) be an approximate identity in $\mathfrak{T}$ satisfying $\| K_j \|_1 \leqslant 1$. Then (2.4.3) gives

$$\| K_j * \mu \|_r \leqslant \kappa r^{1/2} \| K_j^\wedge \mu^\wedge \|_a \leqslant \kappa r^{1/2} \| \mu^\wedge \|_a , \tag{2.4.4}$$

the last term being independent of j. Assuming (as we may with no loss of generality) that $r > 1$, and using the weak compactness of closed balls in L^r, it follows from (2.4.4) that $\mu \in L^r$ and that (2.4.1) holds.

To prove the equivalence of (i) and (ii), we first assume (i). Let $r \in [1, \infty)$, $f \in \mathfrak{T}_E$, and $g \in \mathfrak{T}$. Then

$$\Big| \sum f^\wedge g^\wedge \Big| = | f * g(e) | \leqslant \| f \|_r \| g \|_{r'} \leqslant \kappa r^{1/2} \| f^\wedge \|_a \| g \|_{r'} .$$

It follows that

$$\Big(\sum_E | g^\wedge |^{a'} \Big)^{1/a'} \leqslant \kappa r^{1/2} \| g \|_{r'} .$$

Changing r into r', we obtain (ii) for $g \in \mathfrak{T}$. This result extends to all $g \in L^r$ by an obvious approximation argument.

Now assume (ii) and let $r \in (1, \infty]$, $f \in \mathfrak{T}_E$, and $g \in L^r$. By Hölder's inequality and (2.4.2), we have

$$|f * g(e)| = \left|\sum_E f^\wedge g^\wedge\right| \leqslant \left(\sum_E |f^\wedge|^a\right)^{1/a} \left(\sum_E |g^\wedge|^{a'}\right)^{1/a'}$$
$$\leqslant \|f^\wedge\|_a \kappa r'^{1/2} \|g\|_r,$$

from which it follows that

$$\|f\|_{r'} \leqslant \kappa r'^{1/2} \|f^\wedge\|_a .$$

Thus (2.4.3) holds and, as already noted, this implies that (i) holds.

Remarks 2.5. (i) It is clear that (2.4.3) is true for $r = 1$ whatever the set $E \subseteq X$ (Hausdorff–Young). Similarly, (2.4.2) is true for $r = \infty$ whatever the set $E \subseteq X$ (Parseval).

(ii) When $p = 1$, $a = 2p/(3p - 2) = 2 = a'$ and Theorems 2.1 and 2.4 combine to give the known result that, if E is Sidon, then $\mathbf{M}_E \subseteq \bigcap_{r<\infty} L^r$ and

$$\|\mu\|_r \leqslant \kappa r^{1/2} \|\mu\| \tag{2.5.1}$$

for all $\mu \in \mathbf{M}_E$ and all $r \in [1, \infty)$. When $1 < p < 2$, Theorems 2.1 and 2.4 no longer combine in this way and we are unable to show even that $L_2^E \subseteq \bigcap_{r<\infty} L^r$. For some partial results in this direction, see Section 6.

For any set S, $\nu(S)$ will denote the cardinal number of S. For the special case of arithmetic progressions in $\mathbb{Z}$, the following corollary was proved in [3, 5.2] and also in [10, Proposition 7].

COROLLARY 2.6. *Suppose that $E \subseteq X$ is p-Sidon. Let $\mathscr{F}$ denote a test family of order M of finite subsets Φ of X, as in* [8, *Section* 3]. *Then*

$$\nu(E \cap \Phi) \leqslant \kappa M^{(1/2)a'}(\log \nu(\Phi))^{(1/2)a} \tag{2.6.1}$$

whenever $\Phi \in \mathscr{F}$ and $\nu(\Phi) \geqslant 3$.

Proof. We adapt the proof of (3.2) in [8], choosing $\Psi = \Phi$ from the outset. Thus, we choose $f \in \mathfrak{T}$ such that

$$\begin{gathered} f^\wedge = 1 \quad \text{on} \quad \Phi, \qquad f^\wedge = 0 \quad \text{outside} \quad \Phi^2\Phi^{-1}, \qquad 0 \leqslant f^\wedge \leqslant 1, \\ \|f\|_1 \leqslant \{\nu(\Phi\Phi^{-1})/\nu(\Phi)\}^{1/2} \leqslant M^{1/2}. \end{gathered} \tag{2.6.2}$$

Define

$$g = \sum_{\chi \in E \cap \Phi} \chi \in \mathfrak{T}_E .$$

By (2.6.2) and Theorem 2.4(i), we have

$$\nu(E \cap \Phi) = \sum_{E \cap \Phi} f^\wedge = \sum_{E \cap \Phi} f^\wedge g^\wedge = f * g(e)$$

$$\leqslant \| f \|_{r'} \| g \|_r \leqslant \| f \|_{r'} \kappa r^{1/2} \| g^\wedge \|_a = \| f \|_{r'} \kappa r^{1/2} (\nu(E \cap \Phi))^{1/a},$$

hence

$$\nu(E \cap \Phi)^{1/a'} \leqslant \kappa r^{1/2} \| f \|_{r'}$$

and so

$$\nu(E \cap \Phi) \leqslant \kappa^{a'} r^{(1/2)a'} \| f \|_{r'}^{a'} . \tag{2.6.3}$$

Also by (2.6.2), $\| f \|_1 \leqslant M^{1/2}$ and

$$\| f \|_2 \leqslant (\nu(\Phi^2 \Phi^{-1}))^{1/2} \leqslant M^{1/2} \nu(\Phi)^{1/2},$$

and so Hölder's inequality shows that for $r \geqslant 2$

$$\| f \|_{r'} \leqslant \| f \|_1^{1-2/r} \| f \|_2^{2/r}$$

$$\leqslant M^{1/2-1/r} M^{1/r} \nu(\Phi)^{1/r} = M^{1/2} \nu(\Phi)^{1/r}. \tag{2.6.4}$$

Combining (2.6.3) and (2.6.4), we obtain

$$\nu(E \cap \Phi) \leqslant \kappa^{a'} r^{(1/2)a'} M^{(1/2)a'} \nu(\Phi)^{a'/r} \tag{2.6.5}$$

provided $r \geqslant 2$. Assuming $\nu(\Phi) \geqslant 3$ and taking $r = 2 \log \nu(\Phi)$, (2.6.5) gives (2.6.1).

Corollary 2.7. *Suppose that $1 \leqslant p < 4/3$ and that E is p-Sidon. Then*

$$\sup\{\min(\nu(\Phi), \nu(\Psi)) : \Phi, \Psi \textit{ finite}, \Phi\Psi \subseteq E\} < \infty. \tag{2.7.1}$$

Proof. This is an adaptation of the proof of Theorem (3.5) in [8]. In the present proof, (a*) denotes Eq. (a) of [8]. Assume the corollary false and proceed to a contradiction in the following way. Construct $\chi_1, \chi_2, \ldots$ and E_n exactly as in [8], so that

$$E_n \subseteq E, \qquad n^2/8 \leqslant \nu(E_n) \leqslant n^2 \quad \text{for} \quad n \geqslant 2. \tag{6*}$$

Write

$$g_n = \sum_{\chi \in E_n} \chi. \tag{5*}$$

By Theorem 2.4(i),

$$\| g_n \|_n \leqslant \kappa n^{1/2} \| g_n\hat{\ } \|_a = \kappa n^{1/2} \nu(E_n)^{1/a}$$

and so, by (6*),

$$\| g_n \|_n \leqslant \kappa n^{(1/2)+2/a}. \tag{2.7.2}$$

Define f_n as in (8*), so that (see [8])

$$\| f_n \|_{n'} \leqslant 4 \quad \text{for} \quad n \geqslant 2 \tag{2.7.3}$$

and

$$f_n\hat{\ }(\chi) \geqslant 1/4 \quad \text{for} \quad \chi \in E_n . \tag{12*}$$

It follows that, on the one hand, (2.7.2) and (2.7.3) give

$$\sum f_n\hat{\ } g_n\hat{\ } = f_n * g_n(e) \leqslant \| f_n \|_{n'} \| g_n \|_n \leqslant 4\kappa n^{1/2+2/a}, \tag{2.7.4}$$

and, on the other hand, from (5*), (12*), and (6*) it appears that

$$\sum f_n\hat{\ } g_n\hat{\ } = \sum_{E_n} f_n\hat{\ } \geqslant \nu(E_n)/4 \geqslant n^2/32. \tag{2.7.5}$$

If $p < 4/3$, then $1/2 + 2/a < 2$, and so (2.7.4) and (2.7.5) are plainly contradictory if n is sufficiently large. ∎

See Remark 5.6(iii) concerning Corollary 2.7.

COROLLARY 2.8. *Suppose that E is p-Sidon. Then*

(i) *if $1 \leqslant p \leqslant r \leqslant 2$ and $s = 2pr/(pr + 2p - 2r)$, then*

$$\mathbf{M}_E * L^r \subseteq L^s; \tag{2.8.1}$$

(ii) *if $1 \leqslant p \leqslant 4/3$, then*

$$\mathbf{M}_E * L^{(1/2)a'} \subseteq \bigcap_{s<\infty} L^s; \tag{2.8.2}$$

(iii) *if $1 \leqslant p \leqslant 6/5$ and $r > 1$, then*

$$\mathbf{M}_E * L^r \subseteq \bigcap_{s<\infty} L^s. \tag{2.8.3}$$

Proof. In view of Remark 2.5(ii), the assertions are true when $p = 1$, and so we may assume henceforth that $1 < p < 2$. We suppose that $\nu \in \mathbf{M}_E$, $f \in L^r$, and $g = \nu * f$; in all cases we may and will assume that $1 \leqslant r \leqslant 2$. Evidently, $g \in L_E^r$.

(i) By Theorem 2.1 and the Hausdorff–Young theorem, $\nu\^{} \in l^{a'}$ and $f\^{} \in l^{r'}$. By Hölder's inequality,

$$l^{\alpha} \cdot l^{\beta} \subseteq l^{\alpha\beta/(\alpha+\beta)},$$

and so

$$g\^{} = \nu\^{}f\^{} \in l^{k},$$

where $k = 2pr'/(2p + (2 - p)r')$. Since $1 \leqslant p \leqslant r \leqslant 2$, it follows that $1 \leqslant k \leqslant 2$. Hence, by the Hausdorff–Young theorem again, $g \in L^{k'}$, and $k' = 2pr/(pr + 2p - 2r) = s$.

(ii) By Hölder's inequality,

$$\sum | g\^{} |^{a} \leqslant \Big(\sum | \nu\^{} |^{at'}\Big)^{1/t'} \Big(\sum_{E} | f\^{} |^{at}\Big)^{1/t} \tag{2.8.4}$$

for $1 < t < \infty$. Now suppose that $1 \leqslant p \leqslant 4/3$ and

$$r = p/(2 - p) = (1/2)\, a'.$$

If we take $t = (3p - 2)/4(p - 1)$, then $at' = a'$ and $at = r'$. Hence, Theorem 2.1 and the Hausdorff–Young theorem combine to show that the right-hand side of (2.8.4) is finite. Appeal to Theorem 2.4 now shows that $g \in L^{s}$ for every $s < \infty$.

(iii) Suppose that $1 \leqslant p \leqslant 6/5$ and $r > 1$. Apply (2.8.4) with $t = 2$, noting that $2a \geqslant a'$. Theorem 2.1 asserts that the first factor on the right-hand side of (2.8.4) is finite; the second factor is finite since Theorem 2.4 shows that (2.4.2) holds with f in place of g. Thus $g\^{} \in l^{a}$, and Theorem 2.4 shows that $g \in L^{s}$ for every $s < \infty$.

3. Λ_q Sets and p-Sidon Sets

It is well-known that Sidon sets are Λ_q sets for all $q \in (1, \infty)$; see [11, (37.10)]. Moreover, the p-Sidon non-Sidon sets that we obtain in Section 5 are also Λ_q sets for all $q \in (1, \infty)$; see [4, Théorème 5, p. 359]. It is an open question whether a p-Sidon set must be a Λ_q set for some (or all) $q \in (1, \infty)$. We give below a reasonably complete answer to the following converse question: Must a Λ_q set $(1 < q < \infty)$ be a p-Sidon set for some p, $1 \leqslant p < 2$?

THEOREM 3.1. *Assume that G is infinite. If $1 \leqslant r < 2$, then X contains a set that is a Λ_q set for all $q \in (1, \infty)$ but is not an r-Sidon set.*

Suppose that X contains a subgroup isomorphic with either

(i) $\mathbb{Z}$; *or*

(ii) $\mathbb{Z}(p^{\infty})$ *for some prime* p; *or*

(iii) $\mathsf{P}^{*\infty}_{n=1}\, \mathbb{Z}(p_n)$ *for primes* p_n *with* $\lim_{n\to\infty} p_n = \infty$.

Then X contains a set that is a Λ_q set for all $q \in (1, \infty)$ and is an r-Sidon set for no r, $1 \leqslant r < 2$.

Proof. By (2.3) of [8], X contains a subgroup isomorphic with a group as in (i), (ii), (iii) or

(iv) $\mathbb{Z}(p)^{\aleph_0 *}$ for some prime p.

It suffices for us to consider the cases where X is equal to a group as in (i), (ii), (iii), or (iv).

Case (i). Here $X = \mathbb{Z}$ and we apply 4.11 of [13] with

$$\phi(N) = (\log N)^{\log\log N}.$$

The resulting set E is seen to be r-Sidon for no $r \in [1, 2)$ by applying Corollary 2.6 to appropriate N-element arithmetic progressions Φ_N. Bożejko and Pytlik [3, 5.3], use 4.11 of [13] to obtain a slightly weaker assertion than we have established here.

Case (ii). This case is handled by making simple changes in the proof of Theorem (6.6) of [8], replacing condition (3) of [8] by the condition

$$p^{n_{j+1}}/\log^{2j^2}(p^{n_{j+1}}) \geqslant (8j^2)^{2j}.$$

Here, of course, we apply Corollary 2.6 above instead of (3.3) of [8].

Case (iii). Here we modify the proof of Theorem (6.3) of [8], replacing condition (1) of [8] by the condition

$$p_n > p_n/\log^{n^2}(p_n) > (2n^2)^n \qquad \text{for all} \quad n.$$

Case (iv). It might seem that the proof of Theorem (5.5) of [8] could be modified so as to produce sets which are Λ_q for all $q \in (1, \infty)$ and not r-Sidon, but closer inspection makes this appear not to be the case when $r > 1$. It seems necessary to adopt a different approach which leads only to the weaker first statement in Theorem 3.1. Here we may assume that $G = \mathsf{P}^{\infty}_{j=1}\, \mathbb{Z}(p)$ and $X = \mathsf{P}^{*\infty}_{j=1}\, \mathbb{Z}(p)$. We regard $\mathbb{Z}(p)$ as a subgroup of $\mathbb{T}$ and so, for each j, the jth projection π_j of G belongs to X. Let $E = \{\pi_1, \pi_2, \ldots\}$.

Now consider a fixed r satisfying $1 \leqslant r < 2$, and select an integer k so large that $k > r/(2 - r)$. As in Bonami [4, Théorème 5, p. 359], let E_k denote the set of all characters of the form $\prod_j \pi_j^{\epsilon_j}$, where $\epsilon_j \in \{-1, 0, 1\}$ and $\sum_j |\epsilon_j| = k$. By Bonami's theorem, E_k is a Λ_q set for all $q \in (1, \infty)$. For integers $m \geqslant k$, let $\Phi_m = \mathsf{P}_{j=1}^{m} \mathbb{Z}(p)$, a finite subgroup of X. Then

$$\nu(E_k \cap \Phi_m) \geqslant m(m-1) \cdots (m-k+1)/k!,$$

since each k-element subset S of $\{1, 2,..., m\}$ gives rise to the element $\prod_{j \in S} \pi_j$ of $E_k \cap \Phi_m$. On the other hand, $\nu(\Phi_m) = p^m$, and so an application of Corollary 2.6 to the sets Φ_m shows that E_k cannot be an r-Sidon set.

Remark 3.2. Theorem 3.1 shows that Λ_q sets need not be p-Sidon sets. Nevertheless, certain Λ_q sets share with p-Sidon sets some of the properties given in Section 2. As a simple example, we first prove

If E is a Λ_a set with $a = 2p/(3p - 2)$, then there is a κ such that

$$\| \nu^\wedge \|_{a'} \leqslant \kappa \| \nu \| \qquad \text{for all} \quad \nu \in \mathbf{M}_E; \tag{3.2.1}$$

i.e., the conclusion of Theorem 2.1 *holds.*

Proof. Observe that $1 < a \leqslant 2$. Since E is a Λ_a set, there exists a κ so that $\|f\|_a \leqslant \kappa \|f\|_1$ for all $f \in L_E^1 = L_E^a$. Now if $\nu \in \mathbf{M}_E$, then by (37.7.v) of [11], ν belongs to L_E^a and so by the Hausdorff–Young inequality we have $\| \nu^\wedge \|_{a'} \leqslant \| \nu \|_a \leqslant \kappa \| \nu \|$. ∎

The next theorem provides a special class of sets that satisfy many of the conclusions in Section 2.

THEOREM 3.3. *Suppose that $E \subseteq X$ is a Λ_q set for all $q \in (1, \infty)$. As in* [4], *for each $q > 2$, let A_q denote the smallest number A such that $\|f_q\| \leqslant A \|f\|_2$ for all $f \in L_E^1 = L_E^q$*; cf. [11, (37.7)]. *Suppose further that $A_q = O(q^{(1/2)k})$ for some positive number k. Then*

(i) *for $p \geqslant \max\{1, 2k/(k+1)\}$, properties* 2.4(i), 2.4(ii), *and* (2.6.1) *hold*;

(ii) *if $k < 2$, then* (2.7.1) *holds.*

Proof. The inequality $\|f\|_q \leqslant A_q \|f\|_2$ and a Hahn–Banach argument show that for $q > 2$ and $g \in L^{q'}(G)$,

$$\Big(\sum_E | g^\wedge |^2 \Big)^{1/2} \leqslant A_q \| g \|_{q'}; \tag{3.3.1}$$

compare the proof of (i) $\Rightarrow$ (ii) in (37.9) of [11].

Let C be a positive number such that $A_q \leqslant Cq^{(1/2)k}$ for all $q > 2$. We prove 2.4(ii) for $p \geqslant \max\{1, 2k/(k+1)\}$. First consider $1 < r < 2$ and $g \in L^r$. By (3.3.1) with $q = r'$, we have

$$\left(\sum_E |g^\wedge|^2\right)^{1/2} \leqslant A_{r'} \|g\|_r \leqslant Cr'^{(1/2)k} \|g\|_r . \tag{3.3.2}$$

We also have

$$\sup_E |g^\wedge| \leqslant \|g^\wedge\|_\infty \leqslant \|g\|_1 \leqslant \|g\|_r . \tag{3.3.3}$$

Since $2 \leqslant 2p/(2-p) = a' < \infty$, Hölder's inequality gives

$$\| \ \|_{a'} \leqslant \| \ \|_2^\alpha \| \ \|_\infty^{1-\alpha},$$

where $\alpha = (2-p)/p$, and so (3.3.2) and (3.3.3) imply that

$$\left(\sum_E |g^\wedge|^{a'}\right)^{1/a'} \leqslant C^\alpha r'^{(1/2)k\alpha} \|g\|_r .$$

Since $p \geqslant 2k/(k+1)$, we have $(1/2)k\alpha \leqslant 1/2$ and hence

$$\left(\sum_E |g^\wedge|^{a'}\right)^{1/a'} \leqslant C^\alpha r'^{1/2} \|g\|_r . \tag{3.3.4}$$

This proves 2.4(ii) for $1 < r < 2$ and $\kappa \geqslant C^\alpha$. For any s satisfying $2 \leqslant s \leqslant \infty$, (3.3.4) trivially implies

$$\left(\sum_E |g^\wedge|^{a'}\right)^{1/a'} \leqslant C^\alpha r'^{1/2} s'^{1/2} \|g\|_s$$

for all r satisfying $1 < r < 2$. Since r can be arbitrarily close to 2, we obtain 2.4(ii) in its entirety with $\kappa = 2^{1/2}C^\alpha$.

Theorem 2.4 assures us that 2.4(i) also holds. To see that (2.6.1) holds, we simply note that the proof of (2.6.1) depends only on 2.4(i).

To prove (ii), set $p = \max\{1, 2k/(k+1)\}$ and note that $1 \leqslant p < 4/3$. By (i), 2.4(i) holds for this value of p. Now observe that the proof of (2.7.1) again depends only on 2.4(i).

Remark 3.4. Let k be any positive integer and consider any set E_k as in [4, Théorème 5, p. 359]. Mme. Bonami shows that $E = E_k$ satisfies the hypotheses of Theorem 3.3. For this set and $p \geqslant 2k/(k+1)$, 2.4(i), 2.4(ii), and (2.6.1) all therefore hold. It is natural to inquire whether the set E_k is, in fact, p-Sidon for some p, say, $p = 2k/(k+1)$. We have been unable to answer this general question, but see the examples of 4/3-Sidon non-Sidon sets that are given in Corollary 5.5.

4. Varopoulos Algebras and 4-Norms

In this section we develop a little machinery in order to construct (in Section 5) examples of sets that are p-Sidon sets if and only if $p \geqslant 4/3$. However, the theorems in this section may be of independent interest. Readers interested only in the applications to p-Sidon sets are invited to follow the detour indicated in Remark 5.6(i).

Notation 4.1. We adopt some notation from [15] and [16], as follows. Let D_1 and D_2 be two discrete spaces, and let $D = D_1 \times D_2$. We write $V = V(D)$ for the tensor algebra $c_0(D_1) \widehat{\otimes} c_0(D_2)$, which we take to be defined as a subalgebra of $c_0(D)$; cf. [14, p. 60] or [11, (42.3) and (42.4)]. A complex-valued function β on D is a multiplier of V if $\beta\alpha \in V$ for all $\alpha \in V$. The set $N(D)$ of all multipliers of V is a Banach algebra with norm $\| \beta \|_N$ denoting the operator norm of the mapping $\alpha \mapsto \beta\alpha$ carrying V into V. The only fact about $N(D)$ that we will need is Lemma 1.1 in [16]:

(i) *A bounded function β on D belongs to $N(D)$ if and only if*

$$S_\beta = \sup_F \| \beta \mid F \|_{V(F)} < \infty,$$

where the supremum is taken over all finite subsets F of D of the form $F_1 \times F_2$ (F_i a finite subset of D_i). Moreover, we have

$$\| \beta \|_N = S_\beta \qquad \text{for} \quad \beta \in N(D). \tag{4.1.1}$$

If D_1 and D_2 are finite and α is a complex-valued function on D, $\| \alpha \|_{BM}$ will denote the norm dual to the V-norm of the linear functional on V generated by α:

$$\beta \mapsto \sum_{x \in D_1} \sum_{y \in D_2} \alpha(x, y)\, \beta(x, y);$$

this is also the norm of the bimeasure generated by α. See [15, p. 25] or [14, p. 81].

Lemma 4.2. *There is an absolute constant K such that if D_1 and D_2 are finite, then*

$$\sum_{x \in D_1} \Big(\sum_{y \in D_2} | \alpha(x, y)|^2 \Big)^{1/2} \leqslant (1/2) K \| \alpha \|_{BM} \tag{4.2.1}$$

for all complex-valued functions α on D.

Proof. As is noted in (6.2.2) of [14], we have

$$\| \alpha \|_{BM} = \sup \left\{ \sum_{x \in D_1} \left| \sum_{y \in D_2} \alpha(x, y)\, u_y \right| : \max_{y \in D_2} | u_y | = 1 \right\}. \qquad (4.2.2)$$

Throughout this paragraph we will suppose that α is real-valued and that x is fixed in D_1. Our aim is to specialise an inequality in Section 6.2 of [14] to our setting. Let $n = 1$ and $K_1 = D_2$, so that the space $K = K_1 \times K_2 \times \cdots \times K_n$ is just D_2. Let μ be the function on K_1 defined by $\mu_y = \alpha(x, y)$. Let $\{X^y : y \in K_1\}$ be a sequence of mutually independent normalised normal real random variables on a probability space Ω, each X^y having zero expectation. Thus each X^y is equidistributed with some normal real random variable X on Ω with $EX = 0$ and $\sigma^2 X = 1$. For any positive number a and real random variable Z on Ω, $Z\langle a \rangle$ will denote the truncated random variable given by $Z\langle a \rangle(\omega) = Z(\omega)$ if $| Z(\omega) | \leqslant a$ and $Z\langle a \rangle(\omega) = 0$ if $| Z(\omega) | > a$. Let $\nu(a)$ be the random variable

$$\nu(a) = \sum_{y \in K_1} \mu_y X^y \langle a \rangle.$$

For each y in K_1, $\nu_y(a)$ will denote the constant random variable $\nu_y(a) = \mu_y = \alpha(x, y)$. The inequality (6.2.7) in [14] asserts in this case that

$$E\, | \nu(a) | \geqslant \{(2/\pi)^{1/2} - \sigma(X - X\langle a \rangle)\} \left(\sum_{y \in K_1} | \nu_y(a) |^2 \right)^{1/2}.$$

That is,

$$\int_\Omega \left| \sum_{y \in K_1} \alpha(x, y)\, X^y \langle a \rangle \right|$$
$$\geqslant \left\{ (2/\pi)^{1/2} - \left(\int_\Omega | X - X\langle a \rangle |^2 \right)^{1/2} \right\} \left(\sum_{y \in K_1} | \alpha(x, y) |^2 \right)^{1/2}. \qquad (4.2.3)$$

Since X belongs to $L^2(\Omega)$, we have

$$\lim_{a \to \infty} \int_\Omega | X - X\langle a \rangle |^2 = 0.$$

We now consider a fixed a so large that

$$C = a^{-1} \left\{ (2/\pi)^{1/2} - \left(\int_\Omega | X - X\langle a \rangle |^2 \right)^{1/2} \right\} > 0.$$

The inequality (4.2.3) then reads

$$\int_\Omega \Big| \sum_{y \in K_1} \alpha(x, y)\, X^y\langle a\rangle/a \Big| \geqslant C \Big(\sum_{y \in K_1} |\alpha(x, y)|^2 \Big)^{1/2}. \tag{4.2.4}$$

Now consider a complex-valued function α on D. We can apply (4.2.4) to $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$; a simple argument then shows that

$$\int_\Omega \Big| \sum_{y \in K_1} \alpha(x, y)\, X^y\langle a\rangle/a \Big| \geqslant (1/2)C \Big(\sum_{y \in K_1} |\alpha(x, y)|^2 \Big)^{1/2}. \tag{4.2.5}$$

We have established (4.2.5) for each $x \in D_1$. Since $| X^y\langle a\rangle(\omega)/a | \leqslant 1$ for all $\omega \in \Omega$ and all $y \in D_2$, it follows from (4.2.2) that

$$\begin{aligned} \|\alpha\|_{BM} &\geqslant \int_\Omega \sum_{x \in D_1} \Big| \sum_{y \in K_1} \alpha(x, y)\, X^y\langle a\rangle/a \Big| \\ &\geqslant (1/2)C \sum_{x \in D_1} \Big(\sum_{y \in D_2} |\alpha(x, y)|^2 \Big)^{1/2}. \end{aligned}$$

Thus (4.2.1) holds with $K = 4/C$.

LEMMA 4.3. *There is an absolute constant K such that, if D_1 and D_2 are finite, then*

$$\|\alpha\|_{4/3} \leqslant K \|\alpha\|_{BM} \tag{4.3.1}$$

for all complex-valued functions α on D.

Proof. Littlewood [12, Section 4, pp. 168–169] proves that if $(a_{mn})_{m,n=1}^\infty$ is a doubly infinite sequence of complex numbers, then

$$\Big(\sum_{m,n=1}^\infty |a_{mn}|^{4/3} \Big)^{3/4} \leqslant \sum_{n=1}^\infty \Big(\sum_{m=1}^\infty |a_{mn}|^2 \Big)^{1/2} + \sum_{m=1}^\infty \Big(\sum_{n=1}^\infty |a_{mn}|^2 \Big)^{1/2}. \tag{4.3.2}$$

(It is easily seen that the constant A given by Littlewood can be taken to be 1.) Littlewood's inequality tells us that

$$\|\alpha\|_{4/3} \leqslant \sum_{x \in D_1} \Big(\sum_{y \in D_2} |\alpha(x, y)|^2 \Big)^{1/2} + \sum_{y \in D_2} \Big(\sum_{x \in D_1} |\alpha(x, y)|^2 \Big)^{1/2}.$$

Now apply Lemma 4.2 twice, once with D_1 and D_2 interchanged. ∎

Since the duals of the norms $\| \ \|_{BM}$ and $\| \ \|_{4/3}$ are the norms $\| \ \|_V$ and $\| \ \|_4$, the next theorem is an immediate consequence of Lemma 4.3.

THEOREM 4.4. *There is an absolute constant K such that, if D_1 and D_2 are finite, then*

$$\| \beta \|_V \leqslant K \| \beta \|_4 \tag{4.4.1}$$

for all complex-valued functions β on D.

THEOREM 4.5. *For any discrete spaces D_1 and D_2, we have $l^4(D) \subseteq N(D)$. Also, we have $\| \beta \|_N \leqslant K \| \beta \|_4$ for all $\beta \in l^4(D)$.*

Proof. By 4.1(i), it suffices to show that

$$\| \beta \mid F \|_{V(F)} \leqslant K \| \beta \|_4$$

for all finite subsets F of D of the form $F_1 \times F_2$. This inequality is implied by Theorem 4.4. ∎

See Remark 5.6(iv) concerning Theorem 4.5.

5. EXAMPLES OF NON-SIDON (4/3)-SIDON SETS

We return to the setting of a compact Abelian group G. In this section we suppose also that G is infinite. We will prove (Corollary 5.5) that X always contains a set that is p-Sidon if and only if $p \geqslant 4/3$.

Notation 5.1. Let D_1 and D_2 be infinite disjoint subsets of X such that $D_1 \cup D_2$ is dissociate; such pairs of sets always exist in view of (37.26) of [11]. Let $D = D_1 \times D_2$ and $E = D_1 D_2 \subseteq X$. The product mapping of D onto E is one-to-one and so we freely identify functions on D with functions on E. In particular, if ϕ and ψ are functions on D_1 and D_2, respectively, then $\phi \otimes \psi$ designates the function on E defined by $\phi \otimes \psi(\chi_1 \chi_2) = \phi(\chi_1)\, \psi(\chi_2)$ for $\chi_i \in D_i$. Also (see 4.1(i)), $N(E)$ consists of all complex-valued functions β on E such that

$$\| \beta \|_N = \sup_F \| \beta \mid F \|_{V(F)} < \infty, \tag{5.1.1}$$

the supremum being taken over all finite subsets $F = F_1 F_2$ of E.

LEMMA 5.2. *For $i = 1, 2$, let F_i be a subset of D_i and $F = F_1 F_2$. If $\phi \in l^\infty(F_1)$ and $\psi \in l^\infty(F_2)$, then $\phi \otimes \psi = \nu^\wedge \mid F$ for some $\nu \in \mathbf{M}(G)$ satisfying $\| \nu \| \leqslant 16 \| \phi \|_\infty \| \psi \|_\infty$.*

Proof. We may suppose that $\|\phi\|_\infty > 0$ and $\|\psi\|_\infty > 0$. Suppose first that ϕ and ψ are real-valued. We define β on the dissociate set $F_1 \cup F_2$ as follows:

$$\beta_{\chi_1} = \phi(\chi_1)/2\|\phi\|_\infty \qquad \text{for} \quad \chi_1 \in F_1 ,$$
$$\beta_{\chi_2} = \psi(\chi_2)/2\|\psi\|_\infty \qquad \text{for} \quad \chi_2 \in F_2 .$$

By (37.14) of [11], there is a measure $\mu \in \mathbf{M}^+(G)$ such that

$$\mu\hat{\ }(\chi_1\chi_2) = \beta_{\chi_1}\beta_{\chi_2} \qquad \text{for} \quad \chi_1 \in F_1 , \chi_2 \in F_2 ,$$

and $\|\mu\| = \mu\hat{\ }(1) = 1$. If $\nu = 4\|\phi\|_\infty\|\psi\|_\infty\mu$, then $\nu\hat{\ } \mid F = \phi \otimes \psi$ and

$$\|\nu\| = 4\|\phi\|_\infty\|\psi\|_\infty . \tag{5.2.1}$$

For complex-valued ϕ and ψ, use the equality

$$\begin{aligned}\phi \otimes \psi = (\operatorname{Re}\phi) \otimes (\operatorname{Re}\psi) + i(\operatorname{Re}\phi) \otimes (\operatorname{Im}\psi)\\ + i(\operatorname{Im}\phi) \otimes (\operatorname{Re}\psi) - (\operatorname{Im}\phi) \otimes (\operatorname{Im}\psi)\end{aligned}$$

together with the results of the past paragraph.

THEOREM 5.3. *If $\beta \in N(E)$, then $\beta = \mu\hat{\ } \mid E$ for some $\mu \in \mathbf{M}(G)$.*

Proof. Consider a finite set $F = F_1F_2$, where $F_1 \subseteq D_1$ and $F_2 \subseteq D_2$. By (5.1.1), we have

$$\|\beta \mid F\|_{V(F)} \leqslant \|\beta\|_N .$$

It follows that

$$\beta \mid F = \sum_{k=1}^{\infty} \phi_k \otimes \psi_k ,$$

where each ϕ_k is a complex-valued function on F_1, each ψ_k is a complex-valued function on F_2, and

$$\sum_{k=1}^{\infty} \|\phi_k\|_\infty \|\psi_k\|_\infty \leqslant \|\beta\|_N + 1.$$

For each k, Lemma 5.2 provides us with a measure ν_k such that $\nu_k\hat{\ } \mid F = \phi_k \otimes \psi_k$ and $\|\nu_k\| \leqslant 16\|\phi_k\|_\infty\|\psi_k\|_\infty$. Let $\mu_F = \sum_{k=1}^{\infty} \nu_k$; then $\mu_F \in \mathbf{M}(G)$, $\|\mu_F\| \leqslant 16(\|\beta\|_N + 1)$ and $\mu_F\hat{\ } \mid F = \beta \mid F$.

Now for each finite $F = F_1F_2 \subseteq E$, select μ_F as in the last paragraph. By ordering the sets F by inclusion we obtain a net (μ_F) in $\mathbf{M}(G)$ which, by Alaoglu's theorem, has a weak-* cluster point μ in $\mathbf{M}(G)$. It follows easily that $\mu^\wedge \mid E = \beta$. ∎

Since $l^4(E) \subseteq N(E)$ by Theorem 4.5, we have the following corollary.

COROLLARY 5.4. *If $\beta \in l^4(E)$, then $\beta = \mu^\wedge \mid E$ for some $\mu \in \mathbf{M}(G)$.*

COROLLARY 5.5. *The set $E = D_1D_2$ is p-Sidon if and only if $p \geqslant 4/3$.*

Proof. Corollary 5.4 and Theorem 1.2 show that E is a (4/3)-Sidon set. Corollary 2.7 shows that E cannot be p-Sidon for $p < 4/3$.

Remarks 5.6. (i) Corollaries 5.4 and 5.5 can be proved using Lemma 5.2 and Theorem 4.4; this would avoid any reference to the spaces $N(E)$ or to complete tensor products over infinite spaces.

(ii) In view of Corollary 5.5, we have $f^\wedge \in l^{4/3}$ for all $f \in C_E$. By using the full force of Theorem 5.3, it can be proved that

$$\sum_{\chi_1 \in D_1} \left(\sum_{\chi_2 \in D_2} |f^\wedge(\chi_1\chi_2)|^2 \right)^{1/2} + \sum_{\chi_2 \in D_2} \left(\sum_{\chi_1 \in D_1} |f^\wedge(\chi_1\chi_2)|^2 \right)^{1/2} < \infty \quad (5.6.1)$$

for all $f \in C_E$. Note that in Littlewood's inequality (4.3.2), it is possible for the right-hand side to be infinite while the left-hand side is finite. Thus (5.6.1) is *a priori* stronger than the conclusion $f^\wedge \in l^{4/3}$.

(iii) The sets E in Corollary 5.5 show that Corollary 2.7 is best possible: (2.7.1) can fail for (4/3)-Sidon sets.

(iv) Theorem 4.5 shows that $l^q(D) \subseteq N(D)$ for $q \leqslant 4$. This inclusion cannot hold for $q > 4$. For if $l^q(D) \subseteq N(D)$ for some $q > 4$, then Theorem 5.3 would imply that $E = D_1D_2$ is a p-Sidon set for $p = q' < 4/3$.

6. FURTHER CONSEQUENCES OF p-SIDONICITY

For the reasons described in Remark 2.5(ii), we have been unable to show that (for example) $L_E^1 \subseteq L^2$ when E is p-Sidon and $1 < p < 2$. This section is concerned with a number of weaker results of this sort. Most of the results are more easily evaluated in case $G = \mathbb{T}$, the circle group, to which we shall pay special attention.

Recall that $1 \leqslant p < 2$, $a = 2p/(3p - 2)$, and

$$b = 2(p - 1)/(2 - p) = a'/p'.$$

LEMMA 6.1. *Suppose that $E \subseteq X$ is p-Sidon, and that $\mathscr{F}$ is a test family of order M of finite subsets Φ of X. Let $r \in (1, \infty)$, $s \in (1, \infty)$, $r \leqslant s$, and let $\Phi \in \mathscr{F}$, $\nu(\Phi) \geqslant 3$. Then*

$$\|f\|_r \leqslant \kappa(r, s, M)\|f\|_1 (\log \nu(\Phi))^{bs'/r'} \tag{6.1.1}$$

for every $f \in \mathfrak{T}_{E\cap\Phi}$, where

$$\kappa(r, s, M) = (\kappa M^b s^{1/2})^{s'/r'}$$

and (as in Sections 1 and 2) κ denotes a number depending on p and E.

Proof. (a) We deal first with the case $r = s > 1$. Write temporarily $c = (3p - 2)/(2 - p)$ and apply Hölder's inequality to obtain

$$\|f^\wedge\|_a \leqslant \Big(\sum_{E\cap\Phi} |f^\wedge|^{ac}\Big)^{1/ac} \Big(\sum_{E\cap\Phi} 1\Big)^{1/ac'} = \|f^\wedge\|_{a'}\, \nu(E \cap \Phi)^{2/p'}.$$

By Theorem 2.1 and Corollary 2.6, this gives

$$\begin{aligned}\|f^\wedge\|_a &\leqslant \kappa\|f\|_1 \cdot \kappa^{2/p'} M^{a'/p'} (\log \nu(\Phi))^{a'/p'}\\ &= \kappa\|f\|_1\, M^b (\log \nu(\Phi))^b.\end{aligned}$$

Finally, use Theorem 2.4 to conclude that

$$\|f\|_r \leqslant \kappa M^b r^{1/2}\|f\|_1 (\log \nu(\Phi))^b, \tag{6.1.2}$$

which is the case $r = s > 1$ of (6.1.1).

(b) To deduce the general version of (6.1.1), choose α so that

$$1/r = \alpha/1 + (1 - \alpha)/s,$$

i.e., $\alpha = (s - r)/r(s - 1)$ and $1 - \alpha = (r - 1)s/r(s - 1) = s'/r'$. By Hölder's inequality,

$$\|f\|_r \leqslant \|f\|_1^\alpha \|f\|_s^{1-\alpha}. \tag{6.1.3}$$

If we now use (6.1.2) with s in place of r, (6.1.3) leads to (6.1.1).

Notation 6.2. In what follows $\mathscr{F}$ denotes a test family of order M of finite subsets of X such that $\nu(\Phi) \geqslant 3$ for every $\Phi \in \mathscr{F}$. Also, if $f \in L^1$ and $\Phi \in \mathscr{F}$, we write

$$E_\Phi f = \min\{\|f - h\|_1 : h \in \mathfrak{T}_{\Phi\cap \operatorname{sp} f}\}.$$

THEOREM 6.3. *Let $E \subseteq X$ be a p-Sidon set, and let $\mathscr{F}$ be as in 6.2. Suppose that $r \in (1, \infty)$, that $f \in L^1_E$, and that there is an increasing sequence $(\Phi_k)_{k=1}^{\infty}$ extracted from $\mathscr{F}$ and a number $t > b/r'$ such that*

$$\sum_{k=1}^{\infty} E_{\Phi_k} f \cdot (\log \nu(\Phi_{k+1}))^t < \infty. \tag{6.3.1}$$

Then $f \in L^r$.

Proof. Choose $s \in (r, \infty)$ so large that $bs'/r' \leqslant t$, and then $h_k \in \mathfrak{T}_{E \cap \Phi_k}$ so that

$$\| f - h_k \|_1 = E_{\Phi_k} f;$$

note that the right-hand side here tends to zero as $k \to \infty$ in view of (6.3.1). Observe also that, since $\Phi_k \subseteq \Phi_{k+1}$,

$$\| f - h_{k+1} \|_1 = E_{\Phi_{k+1}} f \leqslant E_{\Phi_k} f.$$

Applying Lemma 6.1 to $h_{k+1} - h_k \in \mathfrak{T}_{E \cap \Phi_{k+1}}$, we obtain

$$\begin{aligned} \| h_{k+1} - h_k \|_r &\leqslant \kappa(r, s, M) \| h_{k+1} - h_k \|_1 (\log \nu(\Phi_{k+1}))^{bs'/r'} \\ &\leqslant \kappa(r, s, M) \cdot 2E_{\Phi_k} f \cdot (\log \nu(\Phi_{k+1}))^t. \end{aligned}$$

It follows from (6.3.1) that $(h_k)_{k=1}^{\infty}$ is a Cauchy sequence in L^r, hence is convergent in L^r; since $h_k \to f$ in L^1, it follows that $f \in L^r$.

The Case $G = \mathbb{T}$. 6.4. For $f \in L^1(\mathbb{T})$ and real θ we write

$$\begin{aligned} \omega_1 f(\theta) &= \int_{\mathbb{T}} | f(e^{i\theta} x) - f(x) | \, d\lambda_{\mathbb{T}}(x) \\ &= (2\pi)^{-1} \int_{-\pi}^{\pi} | f(e^{i\theta + i\phi}) - f(e^{i\phi}) | \, d\phi; \end{aligned}$$

and if $\delta > 0$, we write

$$\Omega_1 f(\delta) = \sup\{\omega_1 f(\theta) : | \theta | < \delta\}.$$

LEMMA 6.5. *Write Ψ_N for the set of all integers j satisfying $| j | \leqslant N$, N denoting any positive integer. If $f \in L^1(\mathbb{T})$, then*

$$E_{\Psi_N} f \leqslant A \Omega_1 f(N^{-1}), \tag{6.5.1}$$

where A is an absolute constant.

We omit the proof of this lemma; cf. Exercises 6.6–6.9 in [7].

COROLLARY 6.6. *Let E be a p-Sidon subset of $\mathbb{Z}$ (regarded as the character group of $\mathbb{T}$), $r \in (1, \infty)$, $f \in L^1_E$. Suppose that for some $t > b/r'$ we have*

$$\omega_1 f(\theta) = O((\log(|\theta|^{-1}))^{-t})$$

for small $\theta \neq 0$. Then $f \in L^r$.

Proof. By (6.6.1) and Lemma 6.5, $E_{\Psi_N} f = O(((\log N)^{-t}))$ for large N. Take $N = N_k = 2^{2^k}$, $\Phi_k = \Psi_{N_k}$, $b/r' < t_1 < t$, and conclude that

$$E_{\Phi_k} f \cdot (\log \nu(\Phi_{k+1}))^{t_1} = O(2^{-k\delta}),$$

where $\delta = t - t_1 > 0$. Thus (6.3.1) is satisfied and the conclusion $f \in L^r$ follows from Theorem 6.3. ∎

We next study relations of the form $f^\wedge \in l^s$ where $s > 0$, dealing first with the case of a general compact Abelian group G. For typographical convenience, we write u for $((2 + s)p - 2s)/2s(2 - p)$ and v for $((2 + s)p - 2s)/2p$. Note that u and v are positive provided that $0 < s < 2p/(2 - p) = a'$.

LEMMA 6.7. *Suppose that E is a p-Sidon subset of X and that $\mathscr{F}$ is as in 6.2. Let $0 < s < 2p/(2 - p) = a'$. Then*

$$\| f^\wedge \|_s \leqslant \kappa'(s, M) \| f \|_1 (\log \nu(\Phi))^u \tag{6.7.1}$$

whenever $\Phi \in \mathscr{F}$ and $f \in \mathfrak{T}_{E \cap \Phi}$; herein

$$\kappa'(s, M) = \kappa^{1+v/s} M^u.$$

Proof. Choose t so that $st = a'$ and apply Hölder's inequality to obtain

$$\| f^\wedge \|_s^s = \sum_{E \cap \Phi} | f^\wedge |^s \leqslant \Big(\sum_{E \cap \Phi} | f^\wedge |^{st} \Big)^{1/t} \Big(\sum_{E \cap \Phi} 1 \Big)^{1/t'}.$$

Applying Theorem 2.1 and Corollary 2.6, it appears that

$$\| f^\wedge \|_s^s \leqslant (\kappa \| f \|_1)^s \, (\kappa M^{(1/2)a'} (\log \nu(\Phi))^{(1/2)a'})^v,$$

from which (6.7.1) follows.

THEOREM 6.8. *Suppose that E is a p-Sidon subset of X, that $\mathscr{F}$ is as in 6.2, and that $0 < s < 2p/(2-p) = a'$. Let $f \in L^1_E$ be such that there is an increasing sequence $(\Phi_k)_{k=1}^{\infty}$ extracted from $\mathscr{F}$ satisfying*

$$\sum_{k=1}^{\infty} E_{\Phi_k} f \cdot (\log \nu(\Phi_{k+1}))^u < \infty. \tag{6.8.1}$$

Then $f^\wedge \in l^s$.

Proof. Choose $h_k \in \mathfrak{T}_{E \cap \Phi_k}$ so that

$$\| f - h_k \|_1 = E_{\Phi_k} f,$$

the right-hand side tending to zero as $k \to \infty$, thanks to (6.8.1). Applying Lemma 6.7 to $h_{k+1} - h_k \in \mathfrak{T}_{E \cap \Phi_{k+1}}$, we obtain

$$\begin{aligned} \| h_{k+1}^\wedge - h_k^\wedge \|_s &\leqslant \text{const.} \| h_{k+1} - h_k \|_1 (\log \nu(\Phi_{k+1}))^u \\ &\leqslant \text{const.} \cdot 2E_{\Phi_k} f \cdot (\log \nu(\Phi_{k+1}))^u . \end{aligned}$$

At this point, (6.8.1) shows that $(h_k^\wedge)_{k=1}^{\infty}$ is a Cauchy sequence in l^s, hence is convergent in l^s. Since $h_k \to f$ in L^1, $h_k^\wedge \to f^\wedge$ pointwise on X, and it follows that $f^\wedge \in l^s$.

Remarks 6.9. (i) Theorems 6.3 and 6.8 may be compared in various ways. If we are concerned with large values of r, it is to be noted that $u \geqslant b$ if and only if $s \leqslant a$; when this is the case, the conclusion $f^\wedge \in l^s$ of Theorem 6.8 combines with Theorem 2.4 to show that $f \in L^r$ for every $r < \infty$. If (6.8.1) holds with $s = 1$, the conclusion of Theorem 6.8 is of course stronger than any relation $f \in L^r$ with $r \in [1, \infty]$.

(ii) We can now specialise Theorem 6.8 to the case $G = \mathbb{T}$, to do which we return to the notation of 6.4 and make use of Lemma 6.5 once more. The result is the following

COROLLARY 6.10. *Let E be a p-Sidon subset of $\mathbb{Z}$,*

$$0 < s < 2p/(2-p) = a',$$

and let $f \in L^1_E(\mathbb{T})$ satisfy

$$\omega_1 f(\theta) = O((\log(|\theta|^{-1}))^{-u} (\log\log(|\theta|^{-1}))^{-\beta}) \tag{6.10.1}$$

for small $\theta \neq 0$, where $\beta > 1$. Then $f^\wedge \in l^s$.

Proof. By (6.10.1) and Lemma 6.5, we have

$$E_{\Psi_N} f = O((\log N)^{-u} (\log \log N)^{-\beta})$$

for large N. Taking $N = N_k = 2^{2^k}$ and $\Phi_k = \Psi_{N_k}$, we conclude that

$$E_{\Phi_k} f \cdot (\log \nu(\Phi_{k+1}))^u = O((\log \log N_k)^{-\beta}) = O(k^{-\beta}),$$

which shows that (6.8.1) holds. It now suffices to appeal to Theorem 6.8.

Remarks 6.11. If (6.10.1) is strengthened to

$$\omega_1 f(\theta) = O(|\theta|^\alpha) \tag{6.11.1}$$

for small $\theta \neq 0$, where $\alpha > 0$, it follows from Corollary 6.10 that $f^\wedge \in l^s$ for every $s > 0$. Actually, however, (6.11.1) implies more than this, as we show in Corollary 6.13 for which purpose we need a lemma.

LEMMA 6.12. *If E is a p-Sidon subset of $\mathbb{Z}$ and $\gamma > 1$, then*

$$\sum_{\substack{n \in E \\ |n| \geqslant 3}} (\log |n|)^{-(1/2)\alpha'} (\log \log |n|)^{-\gamma} < \infty. \tag{6.12.1}$$

Proof. Write $N_k = 2^{2^k}$ and $E_k = \{n \in E : N_k \leqslant |n| < N_{k+1}\}$ and apply Corollary 2.6 to obtain

$$\begin{aligned} &\sum_{n \in E_k} (\log |n|)^{-(1/2)\alpha'} (\log \log |n|)^{-\gamma} \\ &\qquad = O((\log N_{k+1})^{(1/2)\alpha'} (\log N_k)^{-(1/2)\alpha'} (\log \log N_k)^{-\gamma}) \\ &\qquad = O(k^{-\gamma}), \end{aligned}$$

which shows that the sum on the left-hand side of (6.12.1) is majorised by const. $\sum_{k=1}^{\infty} k^{-\gamma} < \infty$.

COROLLARY 6.13. *Suppose that E is a p-Sidon subset of $\mathbb{Z}$ and that $f \in L^1_E(\mathbb{T})$ satisfies* (6.11.1) *for small θ, where $\alpha > 0$. Then*

$$\sum_{n \in \mathbb{Z}} \psi(|f^\wedge(n)|) < \infty \tag{6.13.1}$$

for every positive increasing function ψ on $[0, \infty)$ satisfying $\psi(0) = 0$ and

$$\psi(t) = O((\log t^{-1})^{-(1/2)\alpha'} (\log \log t^{-1})^{-\gamma}) \tag{6.13.2}$$

for small $t > 0$ and some $\gamma > 1$.

Proof. From (6.11.1), it follows (see 2.3.7 of [7]) that $f^\wedge(n) = O(|n|^{-\alpha})$ for large n. Hence, for some $A > 0$ and all large n, we have

$$\begin{aligned}\psi(|f^\wedge(n)|) &\leqslant \psi(A^\alpha |n|^{-\alpha}) \\ &= O((\log |n|)^{-(1/2)\alpha'} (\log\log |n|)^{-\gamma}),\end{aligned}$$

at which stage we appeal to Lemma 6.12.

Remark 6.14. Condition (6.11.1) holds with $\alpha = 1$ if f is of bounded variation; see [1, Vol. 1, p. 40]. Even in the (1-)Sidon case, Corollaries 6.10 and 6.13 seem to be new.

Note added in proof. G. W. Johnson and Gordon S. Woodward (On p-Sidon sets, *Indiana Univ. Math. J.*, to appear) have recently generalised Corollary 5.5 as follows. Let $D_1, D_2, \ldots, D_n$ be disjoint infinite subsets of X whose union is dissociate. Then $D_1 D_2 \cdots D_n$ is p-Sidon if and only if $p \geqslant 2n/(n+1)$. To accomplish this, they find suitable generalisations of 2.7, 4.2 and 4.3.

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